Interpolation with Splines and FFT in Wave Signals

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Abstract. In the digital measurement systems, the sampling frequency is essential by their influence in the measurement's accuracy. Their effect is decisive when it is necessary to obtain the signal pick values. The methods to improve the quality of the acquired data are related with system operation in real time or if it is admitted to carry out samples taking and later on the data analysis.
In this paper, the relationship between the sampling frequency and measurement's maximum error is obtained to sinusoidal continuous signals. This relationship can be extended for signals where one or several fundamental harmonic are outstanding.
By means of an analytic procedure, the mathematical expressions are obtained and maximum error is determined for he picks values measurement. The data are processed by means of the Fast Fourier Transform (FFT) and it is also used, a cubic correction (interpolation of cubic splines).

1 Introduction

At present, the wide use of computerized means in measuring, processing and control systems requires the analysis of the main factors affecting the quality of the information acquired.

The methods used for reducing the effect of a relatively low sampling frequency, without the need to increase it in the data acquisition system, will depend on whether the signal samples require processing while being acquired, or if it is permitted to take some samples, and then process them.

This paper intends to be useful from the academic as well as from the investigating point of view. Its practical value is based on the following reasons:

1. It enables you to calculate, by means of mathematical expressions, the maximum error in computerized measurements of a signal harmonics due to the sampling frequency which influences in the total maximum error of measurements. Such error is also influenced by the sensor's accuracy, which you cannot act upon, and the quantification error of the analogical/digital converter. The latter may be reduced to negligible values with 12- and 16-bit A/D converters.
2. It is possible to choose a sampling frequency that does not represent a heavy burden for the data acquisition system, that is, just the one that is strictly necessary, since it can be programmed on an operative system that is not necessarily a real-time one like Windows or LINUX. This often occurs on using personal computers, which is a tendency for computerized measuring systems in laboratories [5]. This implies an efficient as well as a rational use of the PC's central processing unit. At this point, the main thing is to make little use of the PC's central processing unit; otherwise, it would

be necessary to use an additional hardware, such as an intelligent data acquisition board (with its own processor and operative system) or devices for setting up a distributed system, which makes it more expensive and complex. It is important to properly plan and program such system, for instance, for every variable to be measured, or for choosing a correct sampling frequency that is neither low nor extremely high. In many applications it is useful to "suitably" reduce the sampling frequency in order to acquire, save, and transmit the information and then interpolate in the receiver. The latter would mean that an equivalent and higher sampling frequency has been used [7].

Shannon's sampling theorem [1], [9] establishes the minimum angular sampling frequency (W_s) to reconstruct a continuous X(t) signal based on the samples taken in T time periods, denoting X*(t) as the X(t) digitalized signal.

Considering that:

$$W_s = 2\pi / T$$

Where:

W_s: sampling frequency in radians per second.

T: sampling period in seconds.

For a sinusoidal signal:

$$X(t) = A\sin(w\,t)$$

being w the angular frequency of the X(t) signal. If W_s is greater than or equal as 2w, then the X(t) continuous signal in time can be reconstructed on the basis of the samples taken by a digital device, just as a data acquisition system using an analogical/digital converter and a PC, as shown in Fig.1.

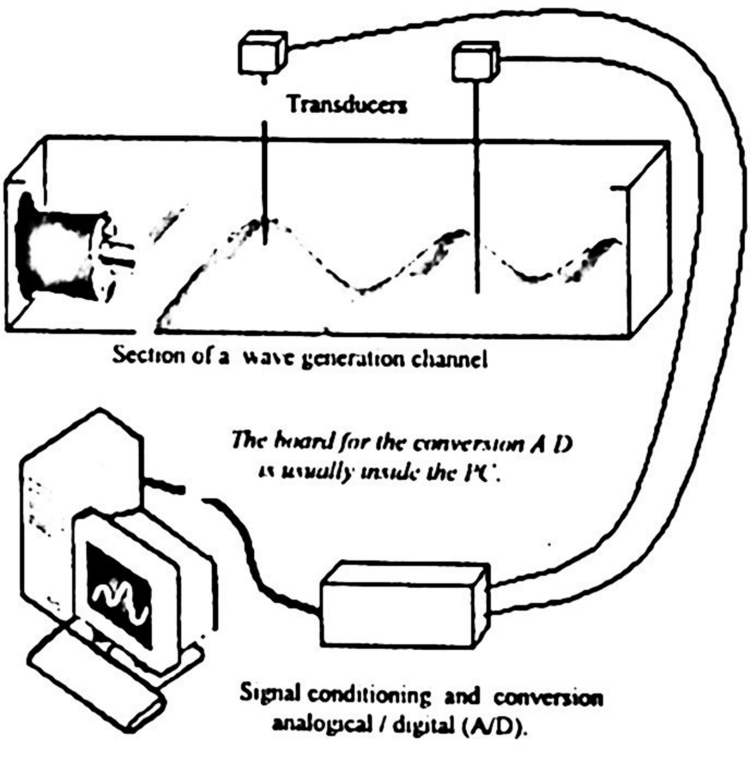


Fig.1. Diagram for measuring waves

For a non-sinusoidal signal, made up of many harmonics, as in the case of an irregular wave signal, the sampling frequency would have to be higher than or the same as twice the highest interest frequency in the spectrum of the signal being measured through its samples, acquired in T periods of time. Being able to reconstruct the continuous signal through its samples does not guarantee the accuracy of this reconstruction. Practical criteria related with the types of applications have been used. For instance, for control systems [1] the sampling frequency should be based on the knowledge of its influence on its own operation. Thus it is reasonable to consider that the highest interest frequency must be closely related to the bandwidth of the closed-loop control system. Therefore, the choice of such sampling frequency must be based on the bandwidth or the rising time of the closed-loop control system. It is adequate to take it between10 and 30 times greater than the bandwidth or choose the sampling period between 4 and 10 times smaller than the rising time, all of which can be small in relation with the criteria to be followed in the typical applications of signal processing [8], [10]. A relatively low sampling frequency in control systems is caused by the fact that its dynamics has low-pass filter characteristics and its typical-time constants are much greater than the closed-loop response time.

Taking into account the primary processing operations set up to improve the accuracy and the destination of the information obtained, there are other practical criteria, such as considering the peak frequency (F_p) of the signal spectrum and taking the sampling frequency (F_s) in accordance with the following relationship:

$F_s \geq 8F_p$ this criterion is recommended for measuring and irregular-wave generation systems in research labs [2], or for taking the sampling period between one tenth or one twentieth of the significant-wave period [4].

Here, F_s has been used to denote the sampling frequency in Hertz (Hz), equivalent to cycles per second or samples per second.

All cases have been based on practical criteria guaranteeing a suitable accuracy depending on the applications, but no mathematical relationships liable to accurately establish the maximum error that might be produced due to the sampling frequency has been stated. This is important when it is necessary to calculate the error of a measuring system which is influenced by all the elements involved, from the continuous-variable sensor up to the analogical/digital converter and the frequency with which the samples are being taken, as shown in Fig.1.

2 Relationship between Sampling Frequency and Error

In digital measuring systems, apart from the quantification error due to the analogical/digital conversion, an error occurs due to the sampling frequency. This error can become extremely serious and should be considered when obtaining the maximum error presented in the measuring.

Next, it is presented an analysis in which a sinusoidal signal is used as an entry, just as it might happen for the generation of regular (sinusoidal) waves, or when it can be considered that there is a fundamental harmonic in the spectrum of the irregular waves. The maximum error of the readings in relation to the sampling frequency when using a zero order hold is determined. This is equivalent to the fact that a sample of the signal keeps its validity right up to the time when the next sample is taken. This maximum error is given by the following expression [1]:

$$E_{maxabs} = \text{Max}\left|x(k+1) - x(k)\right| \quad (1)$$

Being:
E_{maxabs}: absolute maximum error of readings.
x(k), x(k+1) : values of the signal in t = kT and t = (k+1)T, respectively.
T: sampling period (time in-between the taking of the samples of the continuous signal).
k: integer value (1,2,3,.....).

From expression (1) it can be interpreted that value X(k) will be the representative value of the signal until the next sample is taken X(k+1). That is why the $\text{Max}\left|x(k+1) - x(k)\right|$ can be considered as the absolute maximum error in real time (for which there only is information at the moment of the current sampling and at previous moments) and it is caused by the sampling frequency of the signal when using a zero order hold.

The maximum error is located symmetrically to the origin of the axis of the coordinates where the highest speed of signal change takes place, as shown in Fig. 2, being E_{maxabs} the absolute maximum error.

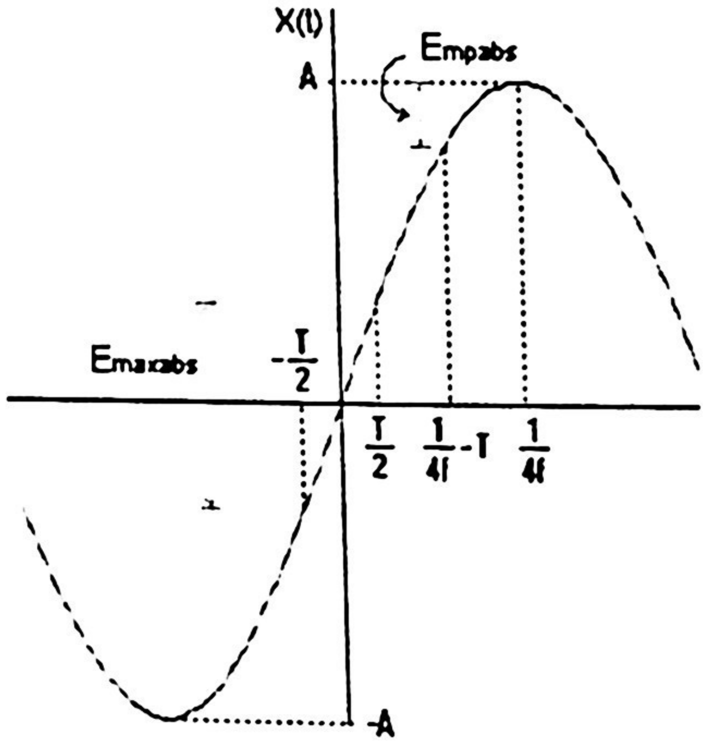


Fig. 2. Absolute maximum error in real time (causal processing) for a sinusoidal signal

E_{mpabs} is the absolute maximum error produced around the peak values in real time or causal processing. From Fig. 2 it can be stated that:

$$X(k+1) = A\sin(2pf\,T/2) \quad (2)$$

$$X(k) = A\sin[2pf(-\ T/2)] \quad (3)$$

Hence:

$$E_{maxabs} - \left|A\sin(2pf\,T/2) - A\sin[2pf(-\ T/2)]\right|$$

$$E_{maxabs} = \left|2A\sin(2pf\,T/2)\right|$$

$$E_{maxabs} = 2A\left|\sin(pfT)\right| \quad (4)$$

Being: $T = 1/F_s$

F_s : sampling frequency. On taking $F_s = nf$, where f is the signal frequency and n an integer greater or equal to two, it is obtained that:

$$E_{maxabs} = 2A\sin(p/n) \quad (5)$$

The maximum relative error, regarding the peak-to-peak value of the signal, expressed in % will be:

$$E_{maxrel} = \frac{2A\sin(\pi/n)}{2A} x100 \quad (6)$$

$$E_{maxrel} = \sin(\pi/n)\,x100 \quad (7)$$

Next, an analysis is presented for formulating the expressions of the maximum error produced around the peak values in causal processing (E_{mpabs}).
The following equation is obtained:

$$X(k+1) = A\sin(2\pi f\,1/4f) = A\sin(\pi/2) = A \quad (8)$$

$$X(k) = A\sin[2pf(1/4f - T)] \quad (9)$$

$$X(k) = A\sin\left[2pf\left(\frac{1 - 4fT}{4f}\right)\right]$$

$$X(k) = A\sin\left[\frac{p}{2}(1 - 4fT)\right]$$

The sampling period can be written as:

$$T = \frac{1}{nf} \quad (10)$$

It is replaced in the previous expression, thus obtaining:

$$X(k) = A\sin\left[\frac{\pi}{2}\left(1 - 4f\frac{1}{nf}\right)\right] \quad (11)$$

$$X(k) = A\sin\left(\frac{\pi}{2} - \frac{2\pi}{n}\right) = A\cos\left(\frac{2\pi}{n}\right) \quad (12)$$

Finally:

$$E_{mpabs} = A - A\cos\left(\frac{2\pi}{n}\right) \qquad (13)$$

and the relative maximum error, regarding the peak-to-peak value of the signal, expressed in % will be:

$$E_{mprel} = \frac{A - A\cos\left(\frac{2\pi}{n}\right)}{2A} \times 100 \qquad (14)$$

$$E_{mprel} = 0.5\left[1 - \cos\left(\frac{2\pi}{n}\right)\right] x100 \qquad (15)$$

In many applications, as in waves analysis, for instance, it is not necessary to make most calculations in real time. Instead, they are to be made on the basis of data previously acquired and determine the maximum and minimum values of the signal, which could be considered as sinusoidal or formed by a main harmonic and other secondary harmonics of small amplitudes. Therefore, in that case, the maximum error that can occur in determining such values is presented in Figure 3 and is called E_{mepabs} as the absolute static maximum error on determining peak values. This is an error in *non-causal processing*, since there is a register of previously acquired information, and then, there is information in previous and subsequent sampling moments for a sampling moment considered as current (k).

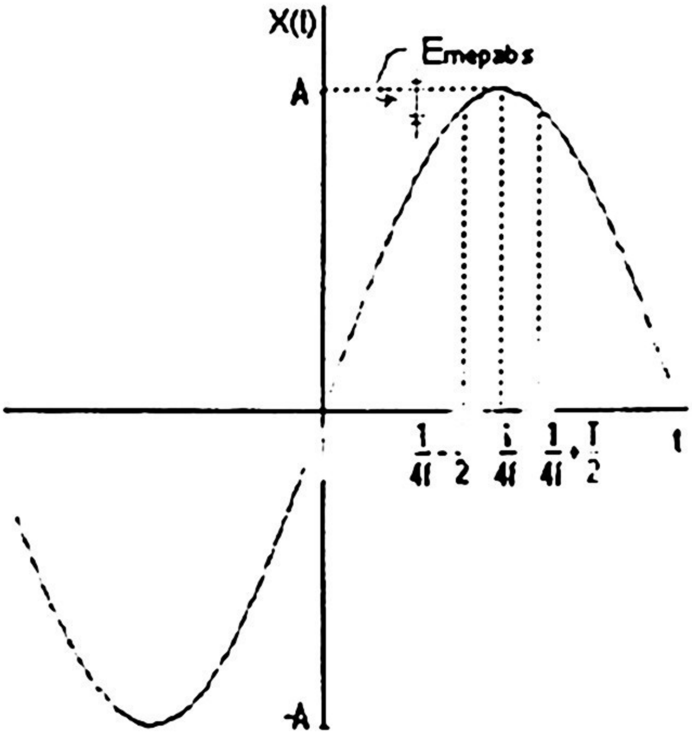


Fig. 3. Absolute static maximum error on determining peak values (non-causal processing)

This error occurs when none of the values of the samples taken coincides with the value of the signal peak. This error will be the maximum when the arrangement of the samples is the one presented in Figure 3; both samples being equidistant from the peak value.

$$E_{mepabs} = |X(k+0.5) - X(k)| \tag{16}$$

Where:

$$X(k+0.5) = A\sin(2\pi f\, 1/4f) \tag{17}$$

$$X(k+0.5) = A\sin(\pi/2) = A \tag{18}$$

$$X(k) = A\sin[2\pi f\, 1/4f - T/2)] \tag{19}$$

$X(k) = A\sin\left[2\pi f\left(\frac{1-2fT}{4f}\right)\right] = A\sin\left[\frac{\pi}{2}(1-2fT)\right]$ Replacing $T = \frac{1}{F_s} = \frac{1}{nf}$; then, the expression is:

$$X(k) = A\sin\left[\frac{\pi}{2}\left(1 - 2f\frac{1}{nf}\right)\right] \tag{20}$$

$$X(k) = A\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = A\cos\left(\frac{\pi}{n}\right) \tag{21}$$

finally, the expression is obtained as:

$$E_{mepabs} = A - A\cos\left(\frac{\pi}{n}\right) \tag{22}$$

and the relative maximum error, regarding the peak-to-peak value of the signal, expressed in % will be:

$$E_{meprel} = \frac{A - A\cos\left(\frac{\pi}{n}\right)}{2A} \times 100 \tag{23}$$

$$E_{meprel} = 0.5\left(1 - \cos\left(\frac{\pi}{n}\right)\right) \times 100 \tag{24}$$

By evaluating expressions (7), (15), and (24), for sampling frequencies, being *n* times the frequency of the sinusoidal signal, the results are presented in Table 1.

Table 1. Comparison of relative errors according to the sampling frequency

n	E_{maxrel} (%)	E_{mprel} (%)	E_{meprel} (%)
2	100	100	50
4	70.71	50	14.64
8	38.26	14.64	3.8
10	30.9	9.54	2.44
20	15.64	2.44	0.61
50	6.27	0.39	0.098
00	3.14	0.098	0.024
200	1.57	0.024	0.0061
300	1.04	0.0109	0.0027

3 Reducing Error Due to a Low Sampling Frequency.

Several methods can be applied in order to recover the continuous signal from the samples, thus improving the accuracy of their reconstruction without needing to raise its sampling frequency considerably. There are various methods, such as:

1. For non-causal processing, in the series of sampled values, the method can interpolate points that are compatible with the complex vector of such series for which the FFT method [13] is used. For instance, when the peak values of the waves are calculated in maritime hydraulics analysis [12].
2. Interpolation with cubic splines [3], [5], [13], for non-causal processing as well.
3. One order hold or higher, for causal processing, which is frequent in real time applications [1].

3.1 Effect of Interpolation using the FFT Method.

The analysis of the effect of the interpolation of points in the temporary series is carried out. These points are compatible with their complex vector in the domain of the frequency. With this method, points are added to a waves register, thus obtaining an effect close to the one that would occur provided it had been sampled at a higher frequency. The number of points to be added can be chosen, calling it filling factor (F_f) and defining it as follows:

$$F_f = \frac{NPFS}{NPOS}$$

Where:

NPFS: number of points of the filled series; NPOS: number of points of the original series. F_f: It must be base power 2 in order to apply Fourier transform algorithm.

Fig. 4 shows part of the water level graph against time for the wave signal. It shows the original series sampled at 3 Hz and the series filled with factor $F_f = 8$.

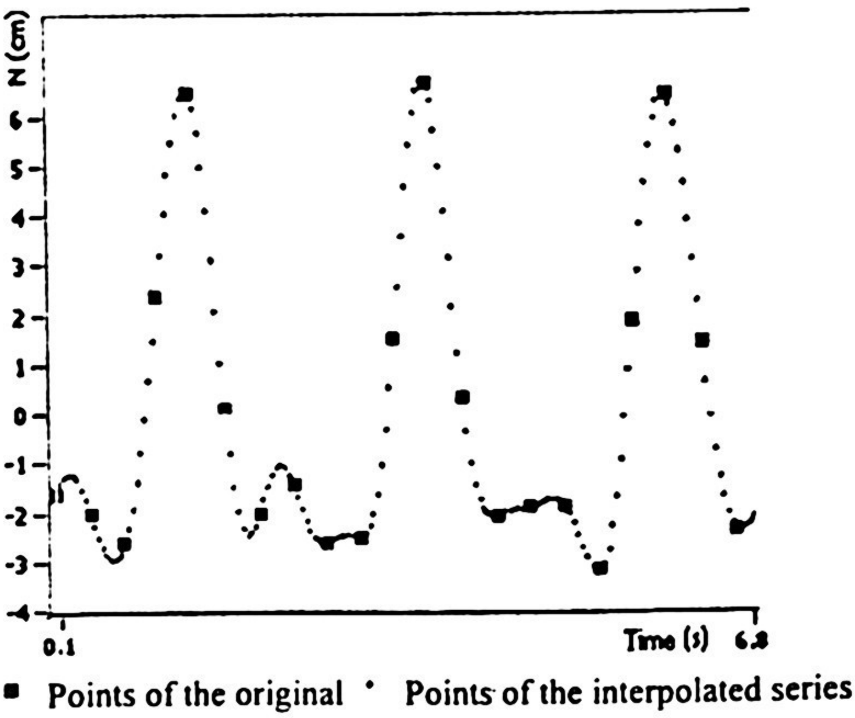


Fig. 4. Example of interpolation with FFT

The original series has been reconstructed with the application of this method, thus counteracting the distortion caused by a low sampling frequency to a great extent. It can be noticed that adding points between the sampled values has made up the signal peaks. Such points keep the compatibility with the complex vector of the original series (in this case, 7 points are added between every two samples).

The waves statistical values are calculated with the interpolated series and the mean and variance of the relative errors are found in respect to the signal sampled at 48 Hz. The summary is shown in Table 2.

It was given the name of interpolated series 1 to those in which 2, 4, 8 and 16 filling factors (F_f) were used for signals sampled with 24, 12, 6 and 3 Hz respectively. For the four signals, the total number of points of the interpolated signals amounted to 4096, which means that the original series sampled at 24 Hz had 2048 points and that the interpolation has only added an intermediate point (between two samples). With this, the series cannot be remade properly. For this signal, the interpolation did not cause a reduction of errors. Errors decrease 4.4, 2.3 and 1.4 times for signals sampled at 12, 6 and 3 Hz, respectively.

Table 2. Summary of the mean and variance of relative errors in %

	24 Hz	12 Hz	6 Hz	3 Hz
Original series	0.48 ± 0.08	1.64 ± 1.52	2.61 ± 3.63	3.1 ± 9.02
Interpolated Series 1	(F_f = 2) 0.48 ± 0.08 *There is not improvement*	(F_f = 4) 0.37 ± 0.13 *It improve 4.4 times*	(F_f = 8) 1.16 ± 0.52 *It improve 2.3 times*	(F_f = 16) 2.28 ± 2.49 *It improve 1.4 times*
Interpolated series 2	(F_f = 4) 0.04 ± 0.002 *It improve 11 times*	(F_f = 8) 0.03 ± 0.001 *It improve 55 times*	(F_f = 16) 0.97 ± 0.76 *It improve 2.8 times*	(F_f = 32) 1.92 ± 3.09 *It improve 1.6 times*

It was given the name of interpolated series 2 to those using 4, 8, 16 and 32 (F_f) factors for the signals sampled at 24, 12, 6 and 3 Hz, respectively. The total number of points of the interpolated series amounted to 8192. For 24 Hz, the interpolation added three intermediate points. 7, 15 and 31 intermediate points were added at 12, 6 and 3 Hz, respectively. In the first two cases (sampling at 24 and 12 Hz, respectively) the improvements are significant, with a decrease of errors between 11 and 55 times and more modest results for the last two signals, with very low sampling frequencies of 6 and 3 Hz, respectively.

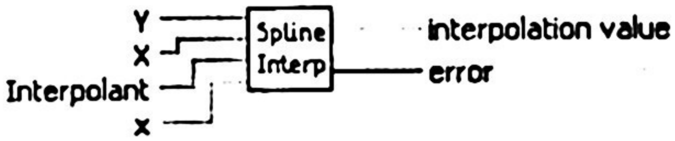
It can be noticed that once the series sampled at 24 Hz and 12 Hz are interpolated, the mean and the variance of relative errors are similar in both cases, and that, for this example, the interpolation makes it unnecessary to sample the continuous signal at 24 Hz, since it is enough to do it at 12 Hz and interpolate with a Ff = 8 filling factor.. Slightly smaller mean-and-variance values for 12 Hz can be observed. It is considered that such behavior is determined by fortuitous factors that mainly depend on the way in which the sampled points are arranged with respect to the continuous signal in time, which, as it is irregular, makes such arrangement un predictable. In tests made elsewhere and not shown in this paper, when adding the same number of intermediate points in both series, the results obtained keep their characteristics.

In all the tests made, the number of waves registered (35 waves) has been kept constant so that only the sampling frequency exerts its influence on the errors of statistical calculations.

3.2 Interpolation with cubic splines (spline interpolation)

For making this interpolation, the mathematics functions of the LabVIEW are used [5]. They are also available in the MATLAB.

According to the notation in the LabVIEW, the following diagram is used:



Spline Interpolation.vi

Where:

Y: arrangement of values to be interpolated.

X: arrangement of the values of axis x; in this case it will be time, which will increase during the sampling period in which the data of arrangement Y were acquired.

x: (small case) value of axis x for which an interpolated point is desired. This value will increase in the new sampling period resulting from interpolation, will be smaller than the original sampling period and will be within the range of values of arrangement X.

Interpolant: second derivative of the cubic spline interpolation function which is calculated by means of another virtual instrument (VI), called ***spline interpolant.vi.***

error: It returns a value that indicates whether the execution of the function has been successful or not.

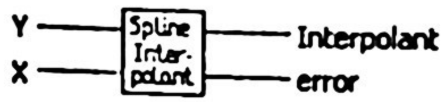
The output value of interpolation z (interpolation value) in interval $[x_i, x_i + 1]$ is given by:

$$z = Ay_i + By_i + 1 + Cy_i'' + Dy_i'' + 1;$$

Where:

$$A = \frac{x_{i+1} - x}{x_{i+1} - x_i},\ B = 1 - A;\ C = \frac{1}{6}\left(A^3 - A\right)\left(x_{i+1} - x_i\right)^2;\ D = \frac{1}{6}\left(B^3 - B\right)\left(x_{i+1} - x_i\right)^2$$

The interpolant value can be obtained with another function (virtual instrument: Spline Interpolant.vi), having registers Z, Y as inputs.



Spline Interpolant.vi

Interpolation function g(x) passes through all points: (x_i, y_i).

$$y_i = g(x_i)\text{ ; where } i = 0,1,\ldots,n-1.$$

The **spline Interpolant.vi** virtual instrument obtains the interpolation function g(x) by interpolating at every $[X_i, X_{i+1}]$ interval with a $P_i(x)$ cubic polynomic function having the following conditions:

1. $p_i(x_i) = y_i$
2. $p_i(x_{i+1}) = y_{i+1}$
3. g(x) has the first and second derivative in interval $[X_0, X_{n-1}]$ continuous. Thus:
   a) $p_i'(x_i) = p_{i+1}'(x_i)$
   b) $p_i''(x_i) = p_{i+1}''(x_i)$

For i = 0, 1,...,n-2.

The following equations are derived from the last condition:

$$\frac{x_i - x_{i-1}}{6} g''(x_{i-1}) + \frac{x_{i+1} - x_{i-1}}{3} g''(x_i) + \frac{x_{i+1} - x_i}{6} g''(x_{i+1}) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$$

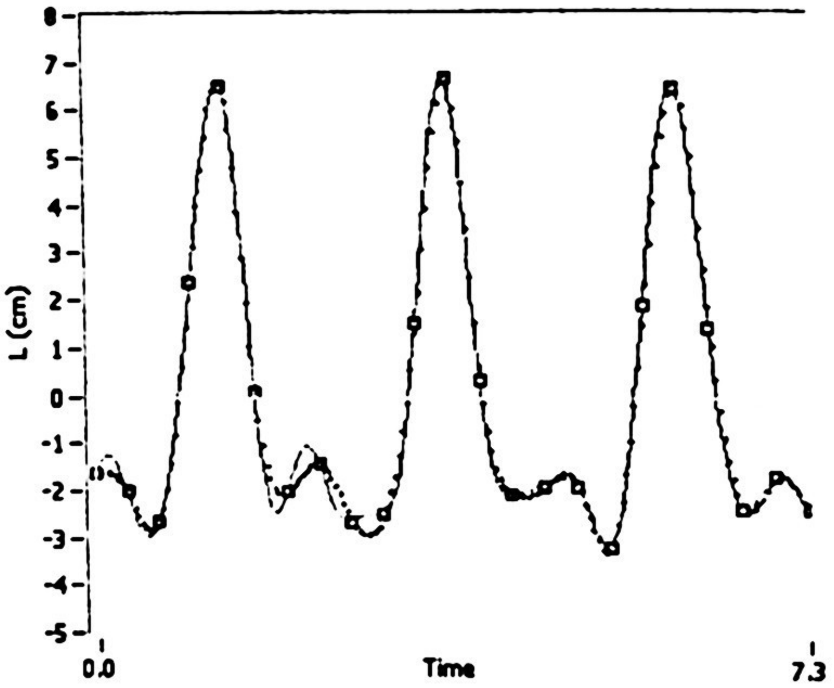
Where i = 1, 2,...,n-2 and n-2 equations are obtained with unknown n $g''(x_i)$, for i = 0, 1,...,n-1. This virtual instrument (spline Interpolant.vi) calculates $g''(x_0)$, $g''(x_{n-1})$ using the following formula:

$$g'(x) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} + \frac{3A^2 - 1}{6}(x_{i+1} - x_i) g''(x_i) + \frac{3B^2 - 1}{6}(x_{i+1} - x_i) g''(x_{i+1})$$

Where:

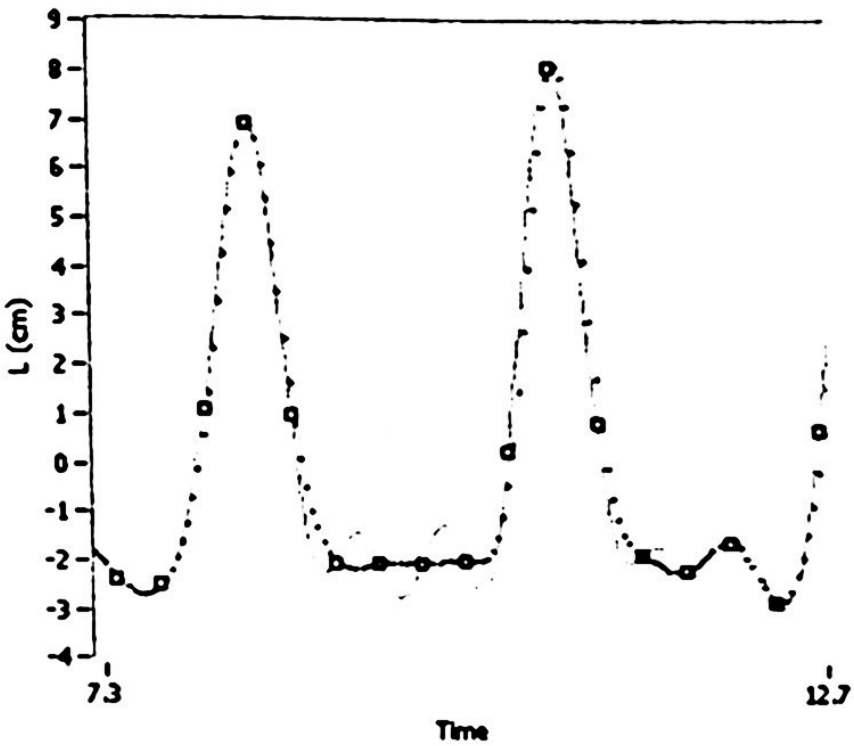
$$A = \frac{x_{i+1} - x}{x_{i+1} - x_i}\text{ ; } B = 1 - A = \frac{x - x_i}{x_{i+1} - x_i}$$

This VI uses $g''(x_0), g''(x_{n-1})$ to solve all the $g''(x_i)$, for i = 1,..., n-2; $g'(x_i)$ is the Interpolant output, which is used as an input in VI *spline interpolation.vi*. Fig. 5 and 6 show two sections of a series interpolated with cubic splines. It also shows the original series and the series interpolated by means of the FFT (complex interpolation) with the goal of facilitating the comparison. Table 3 shows the correlation coefficients. The second column (24Hz and 24Hz_FFT) is the correlation coefficient between the signal sampled at 24 Hz and the interpolated one by using Fourier Fast Transform (FFT) in order to obtain a 24 Hz-equivalent sampling frequency, similarly for Columns 3 and 4. In both interpolations, the original series had been sampled at 3 Hz. Due to that, 7 points were inserted between every two samples to obtain a 24 Hz-equivalent sampling frequency. In the figures presented, you can see that the complex interpolation (FFT) achieves a relatively better reconstruction for this type of wave signal, which has harmonics with small amplitudes and higher frequencies.



■ Points of the original series. • Points of the interpolated series with spline interpolation.
⌒ Points of the interpolated series with FFT.

Fig. 5. Interpolation with cubic splines (a section of the series)



▪ Points of the original series • Points of the interpolated series with spline interpolation.
⌒ Points of the interpolated series with FFT.

Fig. 6. Interpolation with cubic splines (a second section of the series)

Table 3. Correlation coefficients

	24Hz and 24Hz_FFT	24Hz and 24Hz_Spline	24Hz_FFT and 24Hz_Spline
Coeff. of correlation	0.9621	0.6899	0.8011

4 Conclusions

Expressions (7), (15) and (24) are obtained. They make it possible to calculate the relative maximum error due to the sampling frequency (F_s) for a continuous and sinusoidal signal. This can be applied to signals where one or more fundamental harmonics are relevant; for instance, for a regular-wave signal or for the fundamental harmonic of the irregular waves. This error can be considered an indicator of accuracy due to the F_s when it has to do with the complete reconstruction of the continuous signal in time X(t) from the samples taken, using a ladder-type reconstruction or a zero order hold. $E_{tnaurel}$ is the accuracy in causal processing (ordinary in real time applications) and can be interpreted as the maximum delay in perceiving the real value of the signal. Likewise, E_{mprel} would be the maximum delay, but in values close to the signal peaks. However, E_{mcprel} represents the relative maximum error that can occur, in non-causal processing, when you want to calculate the peak values of the signal.

For waves-generation channels with similar characteristics to the one studied, the highest interest frequency in its spectrum is lower than 2 Hz. Therefore, a minimum F_s

between 18 and 20 Hz (samples per second) can be taken, which is equivalent to about 10 samples for each cycle (or wavelength) of the highest-frequency harmonic. Interpolation with cubic splines, widely mentioned and well accepted by many authors and available in signal-processing commercial software, reconstructs the series in the same way as when the Fourier Fast Transform is used. However, with the latter you can achieve a more accurate reconstruction for signals of irregular waves or similar characteristics. This is compared graphically and using the correlation coefficients.

The number of waves registered has been kept constant so that only the F_s exerts its influence on the errors from statistical calculations. The greater the number of waves (many authors recommend about 120), the higher the probability of reducing error in statistical calculations, since the latter are based on the use of the largest waves, which can be detected with a greater degree of probability when a high number of samples is taken.

The results obtained are useful in that they make it possible to assess the contribution of the F_s to the total error of the measuring system (from the transductor to the PC) and to compare two methods widely used in non-causal processing and that can easily be used with any commercial software. Their limitations are related to the use of a sinusoidal signal for making the analysis.

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